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NEW QUESTION.

36. A number of Discussions have been published in this department relating to cubic and biquadratic equations (cf. Vol. XXXV, p. 29, pp. 268–269 and 343–347; Vol. XXIV, pp. 136–137 and 436–439; Vol. XXIII, pp. 314–315). Below are given a number of questions sent in by Professor Harris Hancock of the University of Cincinnati which relate to the cubic and biquadratic and might, perhaps, more properly be proposed as problems were it not for the advantage to be gained, if possible, by treating them all, or at least several of them, in one discussion.

1. For what values of n can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$ where a, b and c are integers?

2. Write the biquadratic in the form $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$. Show that its reducing cubic may be expressed by means of a determinant of the third order which when expanded is

$$4y^3 - g_2y - g_3 = 0,$$

where $g_2 = ae - 4bd + 3c^2$, and

$$g_3 = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = ace - ad^2 - eb^2 - c^3 + 2bcd.$$

3. For the same biquadratic show that

$$a^3(x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2) = 32(3abc - a^2d - 2b^3)$$

without making any use of symmetric functions.

4. If x_0, x_1, x_2 are the roots of a cubic and D its discriminant, show that x_1 is a rational function of x_0 and $\sqrt{D}$. Derive a much simpler relation than that given in Serret, *Cours d'Algèbre Supérieure*, 5th ed., No. 511.

5. If x_0, x_1, x_2, x_3 are the roots of a biquadratic, D its discriminant, e_1, e_2, e_3 the roots of the reducing cubic, show that x_1 is a rational function of x_0, e_1, e_2, e_3 and consequently also of x_0, e_1 and $\sqrt{D}$.

6. If the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0 \tag{1}$$

has a double root, show that the reducing cubic (cf., for example, Burnside and Panton's *Theory of Equations*)

$$t^3 + 3Ht^2 + \left(3H - \frac{a^2I}{4}\right)t - G^2/4 = 0 \tag{2}$$

has a root in common with the cubic

$$8t^3 + 12Ht^2 + G^2 = 0; \tag{3}$$

and conversely, if (2) and (3) have a common root, then (1) has a double root.

DISCUSSIONS.

Concerning Haversines in Plane Trigonometry.

By G. W. Evans, Charlestown High School, Boston, Mass.

The work for the S. A. T. C. in American colleges has called attention to the use of haversines. It appears not to be generally known that plane trigonometry can be simplified, much as to teaching and somewhat as to computation, by the use of these ratios.

The haversine is defined as half the versed sine; that is,

$$\operatorname{hav} A = \frac{1 - \cos A}{2} = \sin^2 \frac{A}{2}.$$

For the case where a right triangle is given by the hypotenuse c and one side, b, nearly equal to c, there formerly appeared in elementary text-books the formula

$$\sin \tfrac{1}{2}A = \sqrt{\frac{c - b}{2c}}$$

but its application seemed to be not much insisted upon. In the form

$$\operatorname{hav} A = \frac{c - b}{2c}$$

it is an obvious deduction from the definition of cos A and there is no square root to be found.

In oblique triangles the case where two sides and the included angle are given and the case where the three sides are given have formulæ that require a considerable amount of detail in the way of proof. With haversines the proofs would be direct and simple.

From the fundamental formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

we obtain

$$\frac{1 - \cos A}{2} = \frac{a^2 - (b - c)^2}{4bc}.$$

From this equation, for the case where two sides and the included angle are given, we have the formula

$$\text{I.} \qquad a^2 - (b - c)^2 = 4bc \operatorname{hav} A.$$

For the case where the three sides are given, we have, from the same equation, the formula

$$\text{II.} \qquad \operatorname{hav} A = \frac{(s - b)(s - c)}{bc}.$$

The following arrangement of work is suggested:

Example 1. Given $A = 94.39°$, $b = 55.12$, $c = 39.90$.

Formulæ: $a^2 - (b - c)^2 = 4bc \operatorname{hav} A$; $\sin B = \dfrac{b \sin A}{a}$.

$A = 94° 23.4'$.		Logarithms.			
$b =$	55.12			b	1.7414
$c =$	39.90			c	1.6010
$b - c =$	15.22		1.1824	hav A	9.7310
$(b - c)^2 =$	231.6		2.3648	4	0.6021
$a^2 - (b - c)^2 =$	4737.				3.6755
$a^2 =$	4969.		3.6963		
$a =$	70.50		1.8482		
$A =$	94.39°	sin A	9.9987	sin A	9.9987
$B =$	51.24°	$\frac{1}{a}$	8.1518	$\frac{1}{a}$	8.1518
$C =$	34.36°	b	1.7414	c	1.6010
Check:	179.99°	sin B	9.8919	sin C	9.7515

Example 2. Given $a = 70.50$, $b = 55.12$, $c = 39.90$.

$$\text{Formula: } \operatorname{hav} A = \frac{(s-b)(s-c)}{bc}.$$

Logarithms.

$a = 70.50$	1.8482			$\frac{1}{a}$	8.1518	$\frac{1}{a}$	8.1518
$b = 55.12$	1.7414	$\frac{1}{b}$	8.2586			$\frac{1}{b}$	8.2586
$c = 39.90$	1.6010	$\frac{1}{c}$	8.3990	$\frac{1}{c}$	8.3990		
$2s = 165.52$							
$s = 82.76$							
$s - a = 12.26$	1.0885				1.0885		1.0885
$s - b = 27.64$	1.4415		1.4415				1.4415
$s - c = 42.86$	1.6321		1.6321		1.6321		
	log hav	A	9.7312	B	9.2714	C	8.9404

Check: 94° 25′ + 51° 13′ + 34° 21′ = 179° 59′.

The work in example 2 very closely resembles the work in spherical trigonometry for the solution of the triangle in which three sides are given. The work in example 1 is of the same character as for the corresponding problem in spherical trigonometry; for example, where the latitude, longitude, and declination are used to compute the altitude for locating a Sumner line. Aside, however, from the advantage that this method of treatment may have in preparing for the work in spherical trigonometry, it has immediate advantage here in the fact that it bases the work in plane triangles on the sine formula for the first two cases, and on the cosine formula for the last two. The use of one or the other of the two haversine formulæ for checking the work in the first two cases makes the check somewhat more laborious than the original computation, but that is no reason why the check should not be insisted upon. On the other hand, the check in the last two cases is simple.

Five place tables of natural haversines and of logarithmic haversines, values for every 15″, are published in Bowditch's *American Practical Navigator* and in publication No. 200 of the U. S. Hydrographic Office; five-place tables for every minute in H. Jacoby's *Navigation* (Macmillan, 1918). The Harvard University Press printed, in the fall of 1918, a four-place table of logarithms of haversines,[1] from which the entries in the above examples are taken. It is to be hoped that such tables will be included in text-books, and made available at the examinations of the College Entrance Board.

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[1] Prepared by Mr. Evans, and occupying two of the four pages of a leaflet.

The term haversine was introduced by James Inman (1776–1859) in the third edition (1835) of his *Navigation and Nautical Astronomy for the Use of British Seamen.* A table of logarithms of haversines is there given. Inman was professor of navigation and nautical science in the Royal Naval College, Portsmouth. See *Dictionary of National Biography.* This note constitutes the reply to Question 2311 (by C. Wargny of Valparaiso) in *L'Intermédiaire des Mathématiciens*, April, 1902.—Editor.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS AND SOLUTIONS TO B. F. FINKEL, Springfield, Mo.

2742. Proposed by S. A. COREY, Des Moines, Iowa.

Establish the following algebraic identity without actually performing the indicated operations:

$$\begin{aligned}
2(t_1t_2 + c_1t_3t_4 + c_2t_5t_6 + c_1c_2t_7t_8)&(r_1r_2 + c_1r_3r_4 + c_2r_5r_6 + c_1c_2r_7r_8) \\
&= (r_1t_1 - c_1r_3t_3 - c_2r_5t_5 + c_1c_2r_7t_7)(r_2t_2 - c_1r_4t_4 - c_2r_6t_6 + c_1c_2r_8t_8) \\
&+ (r_1t_2 - c_1r_3t_4 - c_2r_5t_6 + c_1c_2r_7t_8)(r_2t_1 - c_1r_4t_3 - c_2r_6t_5 + c_1c_2r_8t_7) \\
&+ c_1(r_1t_3 + r_3t_1 - c_2r_5t_7 - c_2r_7t_5)(r_2t_4 + r_4t_2 - c_2r_6t_8 - c_2r_8t_6) \\
&+ c_1(r_1t_4 + r_3t_2 - c_2r_5t_8 - c_2r_7t_6)(r_2t_3 + r_4t_1 - c_2r_6t_7 - c_2r_8t_5) \\
&+ c_2(r_1t_5 + c_1r_3t_7 + r_5t_1 + c_1r_7t_3)(r_2t_6 + c_1r_4t_8 + r_6t_2 + c_1r_8t_4) \\
&+ c_2(r_1t_6 + c_1r_3t_8 + r_5t_2 + c_1r_7t_4)(r_2t_5 + c_1r_4t_7 + r_6t_1 + c_1r_8t_3) \\
&+ c_1c_2(r_1t_7 - r_3t_5 + r_5t_3 - r_7t_1)(r_2t_8 - r_4t_6 + r_6t_4 - r_8t_2) \\
&+ c_1c_2(r_1t_8 - r_3t_6 + r_5t_4 - r_7t_2)(r_2t_7 - r_4t_5 + r_6t_3 - r_8t_1).
\end{aligned}$$

By assuming special relations between the constants involved show that the product of the sum of four squares by the sum of four squares equals the sum of four squares.

2743. Proposed by DANIEL KRETH, Wellman, Iowa.

In the right angle triangle ABC, right angle C, we have given on the hypotenuse the segments $AD = 15$, $DE = 10$, and $EB = 15$; and the angle DCE equal to the angle ECB. Find the angle DCE, and the sides AC and BC.

2744. Proposed by J. B. REYNOLDS, Lehigh University.

The vertices of a triangle are $(0, 0)$, $(2a, 0)$, and $(2x, 2y)$. Where are the vertices of the triangle of least area having its vertices on the perpendicular bisectors of the sides of the given triangle and the same center of gravity as the given triangle.

2745. Proposed by C. N. SCHMALL, New York City.

In the parabola, $y^2 = 4ax$, two normals to the curve are drawn at the ends of a focal chord. Show that the area between these normals and the curve is $20a^2/3 \sin^3 2\phi$, where ϕ is the angle between one of the normals and the x-axis.

2746. Proposed by A. CAMPBELL, St. Johnsbury, Vt.

Given the base, the sum of the sides of the triangle, and the difference of the base angles, to construct the triangle.

2747. Proposed by ENOS W. WITMER, Sophomore in Franklin and Marshall College.

Investigate the problem of solving the equation, $x^4 + ay^4 = w^2 + av^2$. Carmichael's *Diophantine Analysis*, problem 18, page 54.

2748. Proposed by ROGER E. MOORE, University of Wisconsin.

Test for convergence, the series $\sum_{n=1}^{\infty} a_n$, in which

$$a_n = \left[\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\right]^2.$$